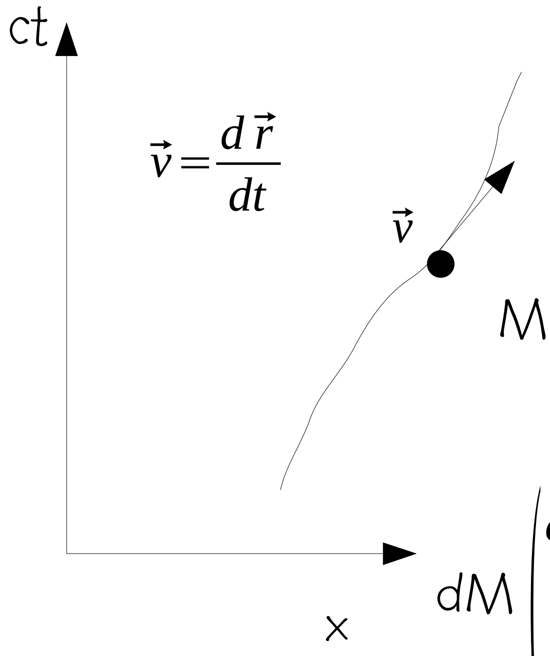
Introduction to Modern Physics : Special Relativity

Lecture 5 : dynamics

4-d velocity : four-velocity of a body



$$\vec{v}=\frac{d\vec{r}}{dt}$$

$$M\begin{pmatrix}ct\\x\\y\\z\end{pmatrix}\qquad dM\begin{pmatrix}cdt\\dx\\dy\\dz\end{pmatrix}$$

$$ds^2=dM^2=c^2dt^2-dx^2-dy^2-dz^2=c^2dt^2-d\vec{r}^{\,2}=c^2dt^2-\vec{v}^{\,2}dt^2=c^2dt^2\left(1-\frac{v^2}{c^2}\right)$$

$$ds^2=\frac{c^2}{\gamma^2}dt^2$$

Comobile IRS is bound to the particle at t. In this IRS, $dt=dt_0$ (proper time) and dx=dy=dz=0.

$$ds^2=\frac{c^2}{\gamma^2}dt^2=c^2dt_0^2 \;\Rightarrow\; dt=\gamma\, dt_0$$

dt_0 is a Lorentz Invariant.

The velocity V in the Minkowski space is defined as : $V = dM / dt_0$

$$V\begin{pmatrix}cdt/dt_0\\dx/dt_0\\dy/dt_0\\dz/dt_0\end{pmatrix}=\begin{pmatrix}\gamma c\\\gamma\, dx/dt\\\gamma\, dy/dt\\\gamma\, dz/dt\end{pmatrix}$$

$$V(\gamma c,\gamma\vec{v})\qquad V^2=\gamma^2(c^2-v^2)=\gamma^2c^2(1-\beta^2)=c^2$$

The modulus of all four-velocities is equal to c which is a constant. In SR, all objects travel with a velocity c but in a four-dimensional space.

four-acceleration

The four-acceleration is defined by :

$$A=\frac{dV}{dt_0}=\left(c\gamma\frac{d\gamma}{dt},\gamma\frac{d\gamma}{dt}\vec{v}+\gamma^2\vec{\phi}\right)$$ where : $\vec{\phi}=\frac{d\vec{v}}{dt}$ is the classical acceleration

As $\frac{d\gamma}{dt}=\gamma^3\frac{\vec{v}\cdot\vec{\phi}}{c^2}$, we finally obtain : $$A=\left(\gamma^4\frac{\vec{v}\cdot\vec{\phi}}{c},\gamma^4\frac{\vec{v}\cdot\vec{\phi}}{c^2}\vec{v}+\gamma^2\vec{\phi}\right)$$

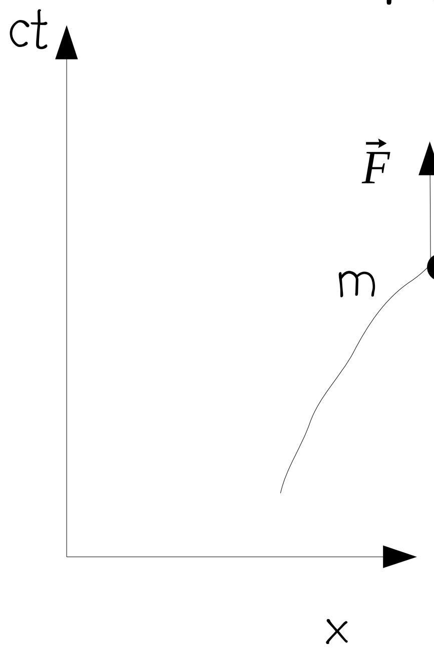
In the comobile IRS : $\vec{v}=\vec{0}$ $A^{com}=(0,\vec{\phi}^{com})$ $(A^{com})^2=A^2=-(\phi^{com})^2$

A four-acceleration is a space-like four-vector. ϕ^{com} is the proper acceleration.

Also : $V\cdot V=V^2=c^2$ then : $\frac{d(V\cdot V)}{dt_0}=\frac{d(V^2)}{dt_0}=0=2V\cdot A=0 \Rightarrow V\cdot A=0$

V and A are orthogonal four-vectors in the Minkowski space.

Fundamental Principle of Dynamics



$$\vec{F}=\frac{d\vec{p}}{dt} \qquad \vec{p}=m\vec{v} \quad \text{classical momentum}$$

Obviously, this equation has to be extended to take into account the fourth dimension of the Minkowski space.

$$K=\frac{dP}{dt_0} \quad \text{with} \quad P=mV \qquad P(\gamma mc, \gamma m\vec{v})=(p_0,\vec{p})$$

four-force — four-momentum — mass (which is a constant !) — four-velocity

Elementary work of the force :

$$dW=\vec{F}\cdot d\vec{r}=\frac{d\vec{p}}{dt}\cdot\vec{v}\,dt=(m\frac{d\gamma}{dt}\vec{v}+m\gamma\vec{\phi})\cdot\vec{v}\,dt=(m\gamma^3\frac{\vec{v}\cdot\vec{\phi}}{c^2}\vec{v}+m\gamma\vec{\phi})\cdot\vec{v}\,dt=m\gamma(\gamma^2\beta^2+1)\vec{v}\cdot\vec{\phi}\,dt=m\gamma^3\vec{v}\cdot d\vec{v}=dT$$

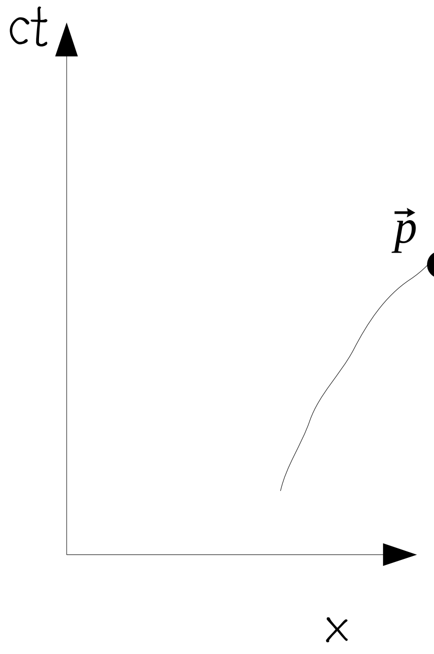
where dT is the elementary variation of the kinetic energy. $\qquad \gamma^2\beta^2+1=\gamma^2$

$dT=d(\gamma mc^2) \Rightarrow T=\gamma mc^2+cte \qquad$ T=0 when $\vec{v}=\vec{0}$ then $T=\gamma mc^2-mc^2=E-mc^2$

$E=T+mc^2$ is by definition the total energy of the body $\qquad E_0=mc^2$

The mass of an isolated system is its total energy observed at rest divided by c^2 .

Four-momentum



$P \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$

$$P^2 = \frac{E^2}{c^2} - \vec{p}^2 = P_0^2 = m^2 c^2$$

$P_0 \begin{pmatrix} m c \\ 0 \end{pmatrix}$ four-momentum in the comobile IRS

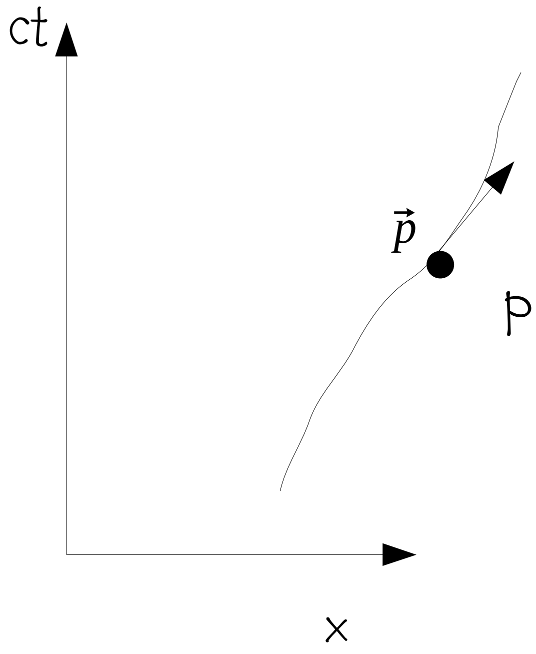
$$E^2 - \vec{p}^2 c^2 = m^2 c^4 \Rightarrow E^2 = \vec{p}^2 c^2 + m^2 c^4$$

Einstein relation

$$P(E/c, \gamma m \vec{v}) = (\gamma m c, \gamma m \vec{v}) \Rightarrow \frac{\vec{v}}{c} = \frac{\vec{p}}{E} c \Rightarrow \beta = \frac{p c}{E}$$

$$\gamma = \frac{E}{m c^2}$$

Kinetic energy



$\begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$

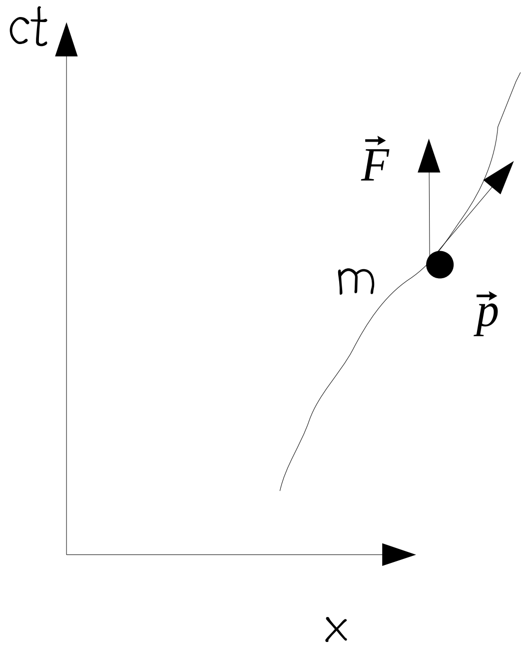
$E = T + mc^2$ T is the kinetic energy

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 = T^2 + m^2 c^4 + 2mc^2 T \Rightarrow T^2 + 2mc^2 T = \vec{p}^2 c^2$$

In the case of low velocities, when : $T \ll mc^2$

$T \simeq \dfrac{\vec{p}^2}{2m}$ which is the classical expression of the kinetic energy

Four-force



$$\vec{F}=\frac{d\vec{p}}{dt} \qquad \vec{p} \quad \text{momentum}$$

$$K=\frac{dP}{dt_0} \qquad \text{with} \qquad P=mV \qquad P\left(\frac{E}{c},\vec{p}\right)$$

four-force

$$\frac{dP}{dt_0}=\left(\frac{\gamma}{c}\frac{dE}{dt},\gamma\frac{d\vec{p}}{dt}\right)=\left(\frac{\gamma}{c}\frac{dT}{dt},\gamma\frac{d\vec{p}}{dt}\right)=\left(\gamma\frac{\vec{F}\cdot\vec{v}}{c},\gamma\frac{d\vec{p}}{dt}\right)$$

$$K=(K_{0,}\vec{K})=\left(\gamma\frac{\vec{F}\cdot\vec{v}}{c},\gamma\vec{F}\right)$$

In the case of low velocities : $K\simeq(0,\vec{F})$

$K^2=-(\overrightarrow{F^{com}})^2$ where $\overrightarrow{F^{com}}$ is the force in the comobile IRS

K is then a space-like four-vector

Isolated systems

If K = (0,0) , the system is isolated :

$$\frac{dP}{dt_0}=0 \Rightarrow \frac{dE}{dt}=0 \text{ and } \frac{d\vec{p}}{dt}=0$$

In SR, the total energy and the total momentum of an isolated system are conserved, i.e. time invariant.

The center-of-mass IRS of an isolated system is defined as the IRS in which its total momentum is equal to zero.

Massless particles

Contrary to classical mechanics, in SR, a massless particle carries a momentum :

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 = \vec{p}^2 c^2 \qquad \text{if } m=0$$

$$|\vec{p}| = p = \frac{E}{c}$$

$$E = \gamma m c^2 \Rightarrow m = \frac{E}{c^2}\sqrt{1-\frac{v^2}{c^2}}$$

then a massless particle is a particle that travels at the limit velocity c

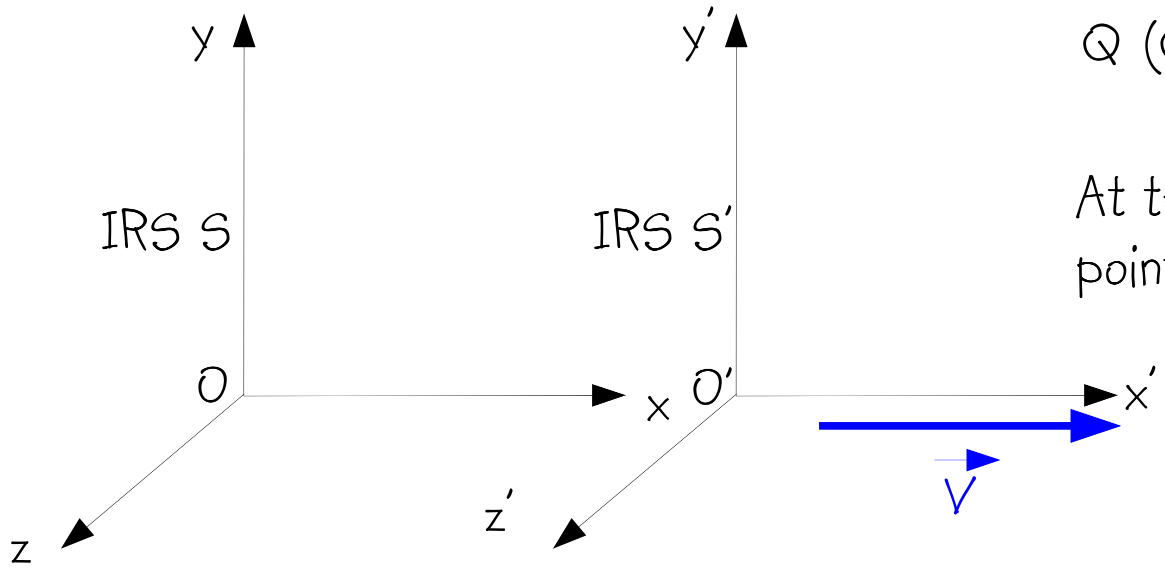
This is the case of photons but this is true for any kind of massless particles (gluons, gravitons ....)

In the case of photons, the total energy is also related to their wave frequency ν :

$$E = h\nu = pc$$

where h = 6,626 10^{-34} J.s is the Planck constant.

Lorentz transformation of four-vectors :



Q (Q_0, Q_x, Q_y, Q_z) coordinates in S

At t=0, O and O' meet at the same point x=0.

$$\begin{pmatrix} Q'_0 \\ Q'_x \\ Q'_y \\ Q'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_x \\ Q_y \\ Q_z \end{pmatrix}$$

All four-vectors transform in the same way , be it an event, a four-momenta, a four-velocity, a four-acceleration, a four-force ....

Four-vectors are not Lorentz invariant, but their moduli are Lorentz invariant !

The scalar product of two four-vectors is also a Lorentz invariant.

To learn more :

-Modern Physics for Scientists & Engineers, Stephen Thornton & Andrew Rex

-Special Relativity : A first encounter , Domenico Giulini, Oxford University Press

-Introduction à la Relativité : Johann Collot, http://lpsc.in2p3.fr/collot

-Relativité restreinte, Claude Semay et Bernard Silvestre-Brac, Dunod