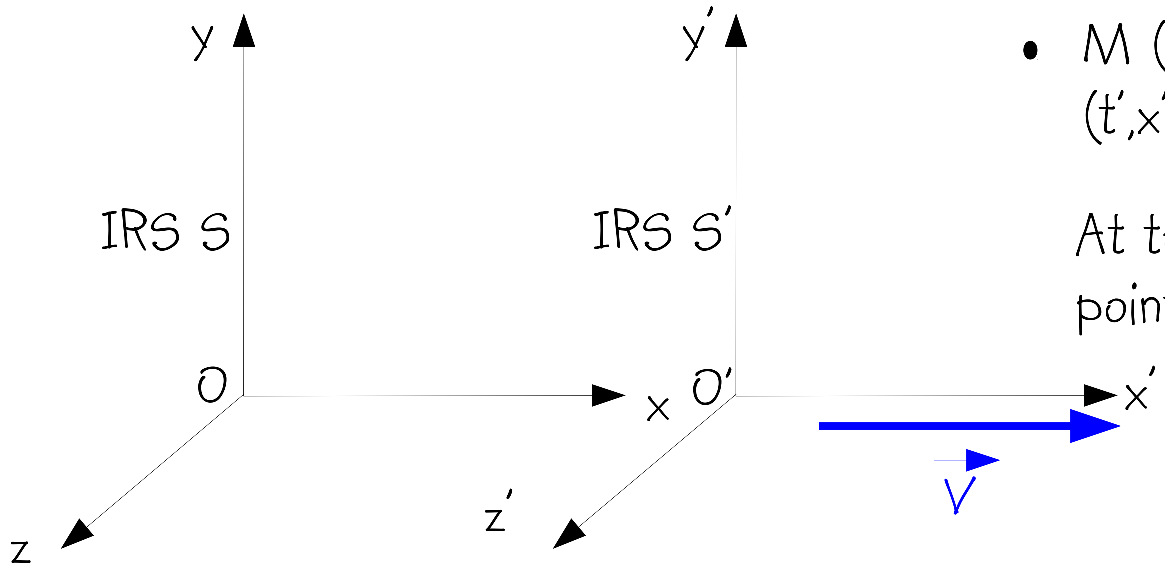
Introduction to Modern Physics : Special Relativity

Lecture 4 : Minkowski's four-dimensional space-time

Hermann Minkowski's opening words during the 80th Assembly of German Natural Scientists and Physicians in September 1908.

The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. Their tendency is a radical one. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve independence.

Lorentz transformation :



- M (t,x,y,z) coordinates in S
  (t',x',y',z') coordinates in S'

At t=0, O and O' meet at the same point x=0.

Relativistic coefficients : $\beta = \frac{V}{c}$ $\quad \gamma = (1-\beta^2)^{-1/2}$

The Lorentz transformation in Cartesian coordinates is given by :

$$ct' = \gamma(ct - \beta x)$$

$$x' = \gamma(x - \beta ct)$$

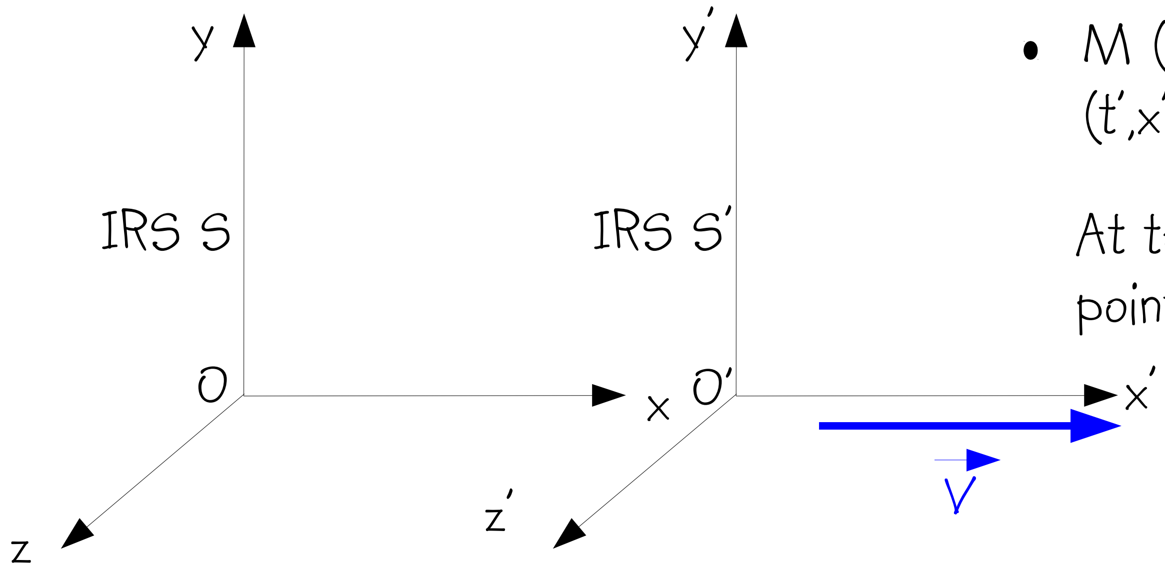
$$y' = y$$

$$z' = z$$

In matrix notation :

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Lorentz transformation :



- M (t,x,y,z) coordinates in S
  (t',x',y',z') coordinates in S'

At t=0, O and O' meet at the same point x=0.

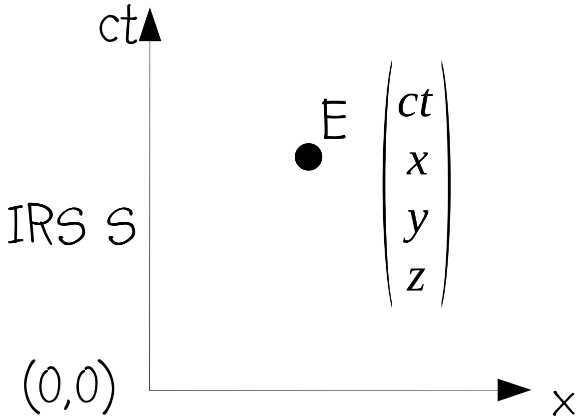
Defining the rapidity as : $\alpha = \text{artanh}\,\beta$ one obtains : $\gamma = \cosh\alpha$; $\beta\gamma = \sinh\alpha$ (exo I)

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

It looks like a rotation that mixes space and time, in fact ct, but as c is constant, ct is a time variable that possesses length units.
Space rotations keep $r^2 = x^2+y^2+z^2$ invariant.
Likewise, LTs keep $s^2=c^2t^2 - r^2$ invariant. (exo I)

s^2 looks like a distance in a four-dimensional space : the Minkowski space. s^2 is called the Minkowskian squared interval, s being the spacetime interval. s^2 is a relativistic invariant.

Minkowski space and four-vectors :



$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Spacetime diagram : only time and x dimensions are represented. The two other space dimensions which are perpendicular to the relative motion are not changed. Consequently, they are not represented.

A point in the Minkowski space is called an event. It is a four-vector : a four dimensional vector.

The Minkowski space is then endowed with a scalar product of four-vectors which defines the squared interval :

$$A\cdot B = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 = a_0 b_0 - \vec{a}\cdot\vec{b}$$

scalar product of four-vectors in Minkowski space

scalar product of space vectors in Euclidean space

$$A \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad B \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \vec{a} \begin{pmatrix} a_1 = a_x \\ a_2 = a_y \\ a_3 = a_z \end{pmatrix}$$

Show that :

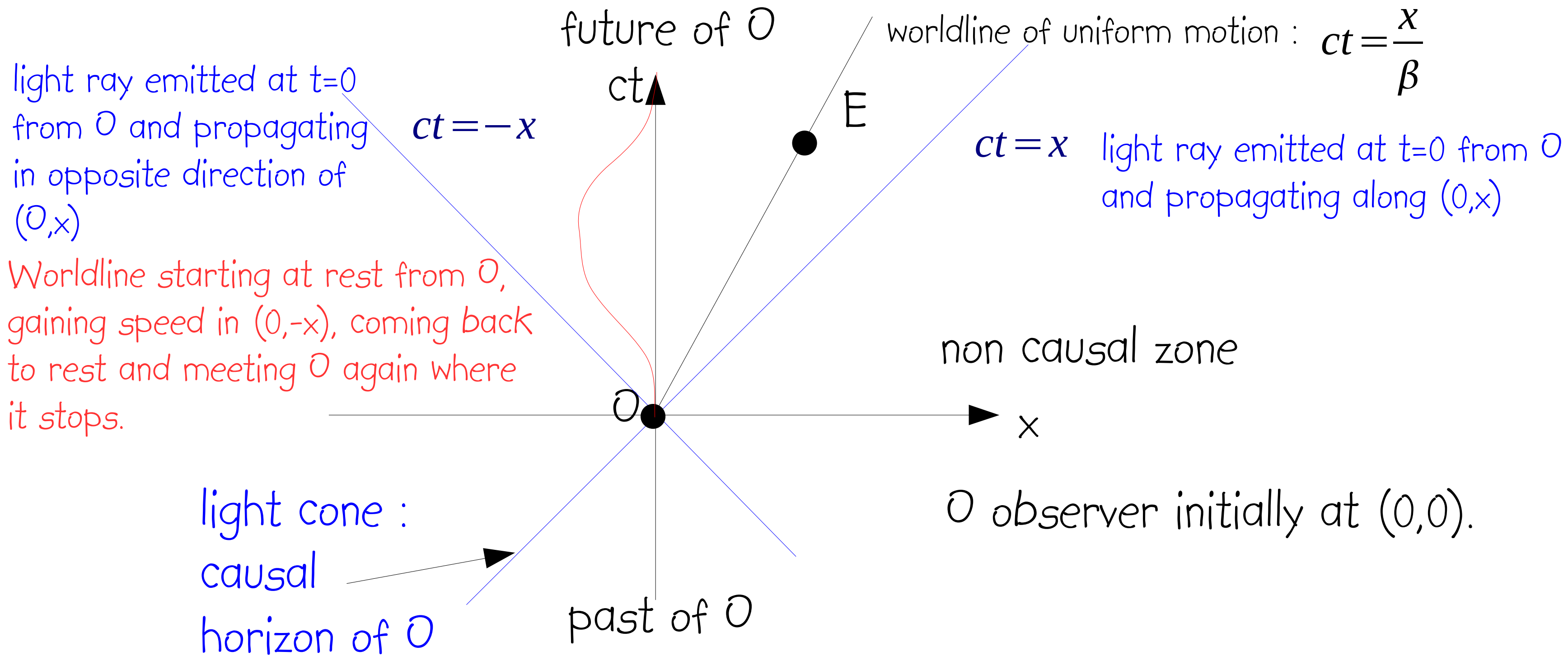
$$(A+B)^2 = A^2 + B^2 + 2\,A\cdot B = a_0^2 - \vec{a}^2 + b_0^2 - \vec{b}^2 + 2 a_0 b_0 - 2\vec{a}\cdot\vec{b} = (a_0 + b_0)^2 - (\vec{a}+\vec{b})^2$$

$$(A-B)^2 = A^2 + B^2 - 2\,A\cdot B = a_0^2 - \vec{a}^2 + b_0^2 - \vec{b}^2 - 2 a_0 b_0 + 2\vec{a}\cdot\vec{b} = (a_0 - b_0)^2 - (\vec{a}-\vec{b})^2$$

$$A^2 = a_0^2 - \vec{a}^2$$

Any scalar product of four-vectors is a Lorentz Invariant (LI). Indeed what is true for scalar products necessarily holds for squared moduli . A^2, B^2 and A.B are LIs.

Minkowski space and interval types :

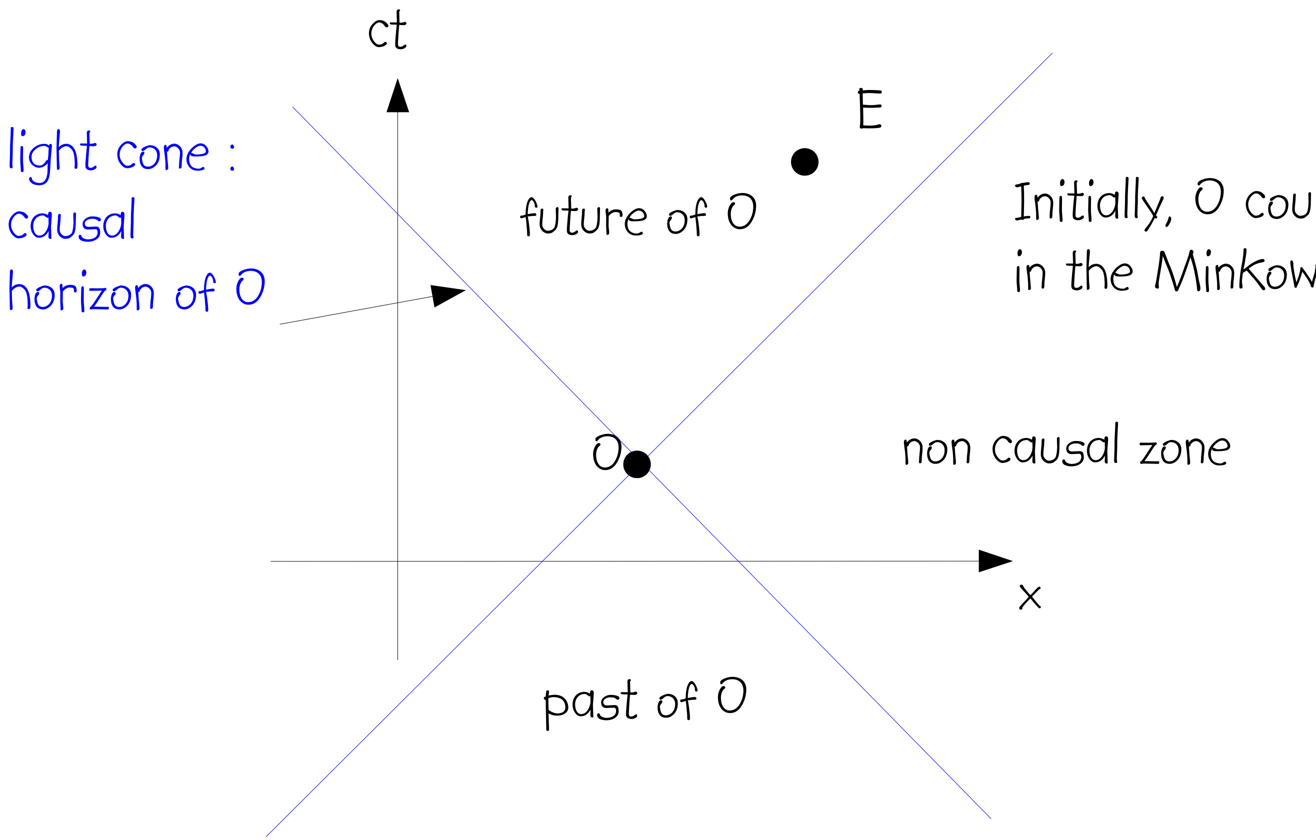


-if $E^2 = s^2 = c^2t^2 - x^2 > 0$, the interval is time-like : O & E may be connected by a physical worldline. O and E are causally connected. In that situation, one may always find an IRS where E is at rest, then x=0 and $t_0^{\,2} = s^2 / c^2$, which is called the proper time.

-If $E^2 = s^2 = c^2t^2 - x^2 < 0$, the interval is space-like : a light ray emitted from O will never reach E. O and E are causally unconnected. In that situation, one may always find an IRS where E is synchronous to O , t remaining = 0, then $x_0^{\,2} = -s^2$, which is called the proper length.

-If $E^2 = s^2 = c^2t^2 - x^2 = 0$, the interval is light-like.

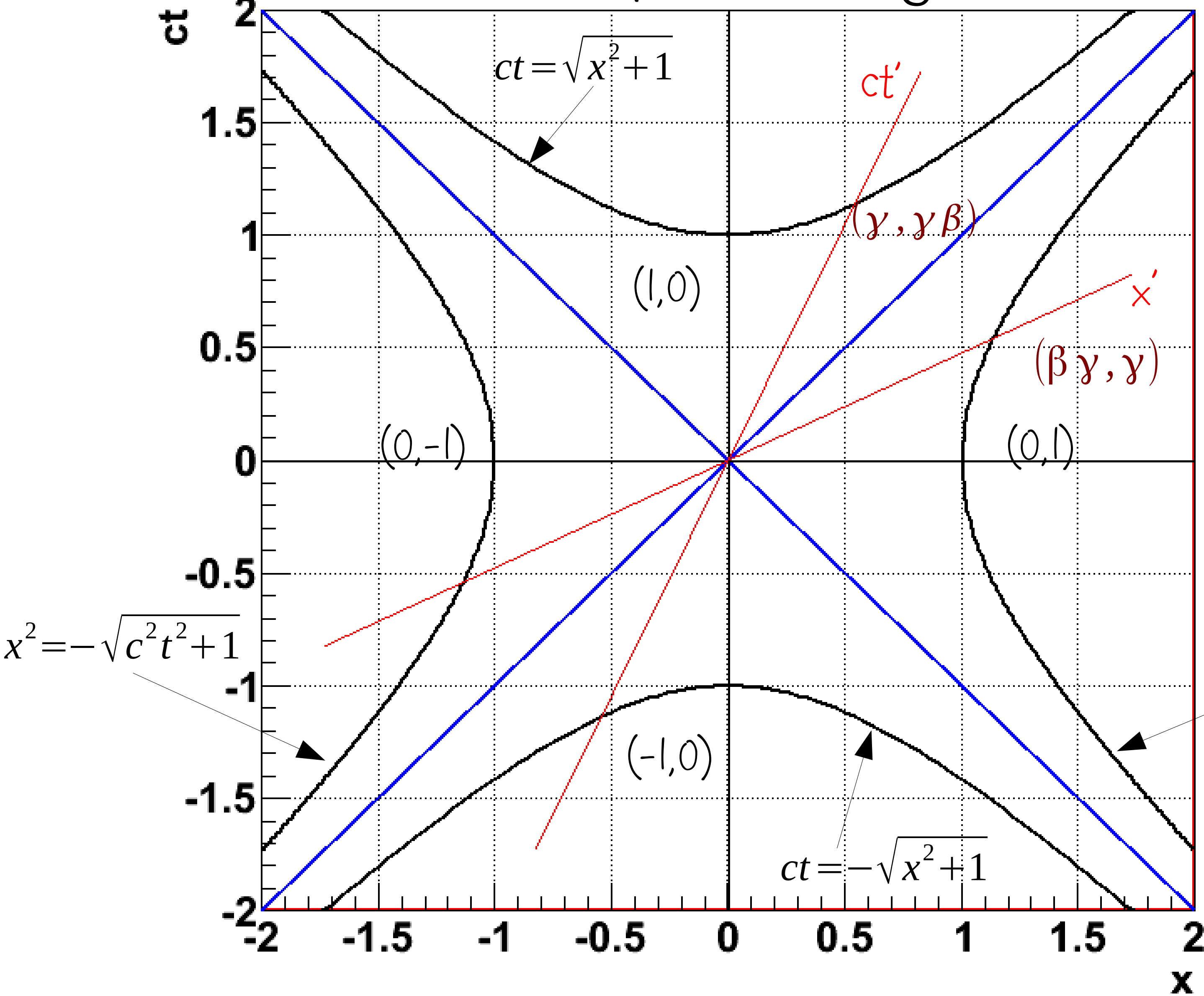
Minkowski space and interval types :



Initially, O could be at any point in the Minkowski space.

Squared interval is given by : $(E-O)^2$

Unit time and unit spatial length Lorentz hyperbolas :



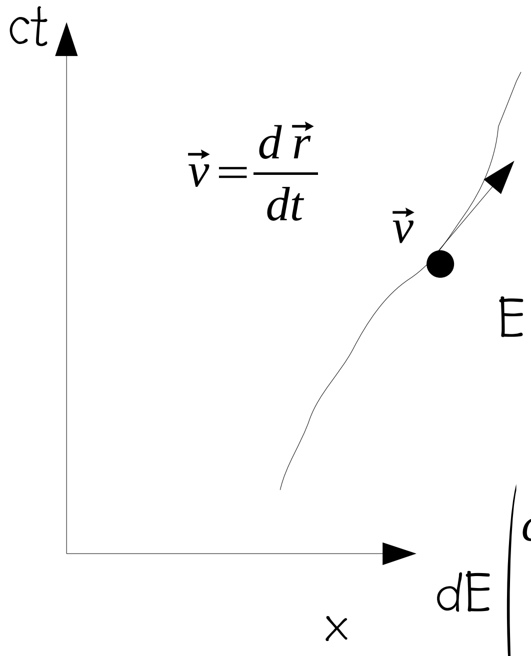
Consequences :

- time-like intervals can have their spatial ordering reversed (symmetry of hyperbola around ct axis) by an IRS change

- space-like intervals can have their time ordering reversed (symmetry of hyperbola around x axis) by an IRS change

Hyperbolas are the images of the unit four-vectors by LTs (change of IRS).

4-d velocity : four-velocity



$$\vec{v}=\frac{d\vec{r}}{dt}$$

$$E\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$dE\begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}$$

$$ds^2=dE^2=c^2dt^2-dx^2-dy^2-dz^2=c^2dt^2\left(1-\left(\frac{d\vec{r}}{dt}\right)^2\frac{1}{c^2}\right)=c^2dt^2\left(1-\frac{v^2}{c^2}\right)$$

$$=\frac{c^2}{\gamma^2}dt^2$$

Comobile IRS is bound to the particle at t. In this IRS, $dt=dt_0$ (proper time) and $dx=0$.

$$ds^2=\frac{c^2}{\gamma^2}dt^2=c^2dt_0^2 \;\Rightarrow\; dt=\gamma\, dt_0$$

dt_0 is a Lorentz Invariant.

The velocity V in the Minkowski space is defined as : dE/dt_0

$$V\begin{pmatrix} cdt/dt_0 \\ dx/dt_0 \\ dy/dt_0 \\ dz/dt_0 \end{pmatrix}=\begin{pmatrix} \gamma c \\ \gamma\, dx/dt \\ \gamma\, dy/dt \\ \gamma\, dz/dt \end{pmatrix}$$

$$V(\gamma c,\gamma\vec{v}) \qquad V^2=\gamma^2(c^2-v^2)=\gamma^2c^2(1-\beta^2)=c^2$$

The modulus of all four-velocities is equal to c which is a constant. In SR, all objects travel with a velocity c but in a four-dimensional space.

To learn more :

-Modern Physics for Scientists & Engineers, Stephen Thornton & Andrew Rex

- Special Relativity : A first encounter , Domenico Giulini, Oxford University Press

-Introduction à la Relativité : Johann Collot, http://lpsc.in2p3.fr/collot

-Relativité restreinte, Claude Semay et Bernard Silvestre-Brac, Dunod

-The principle of relativity, Dover, New York

-Albert Einstein's special theory of relativity, A. Miller, Springer

-The Feynman lectures on physics, volume 1 , Addison-Wesley