Introduction to Modern Physics : Special Relativity

Lecture 2 : Physical implications
Lorentz transformation

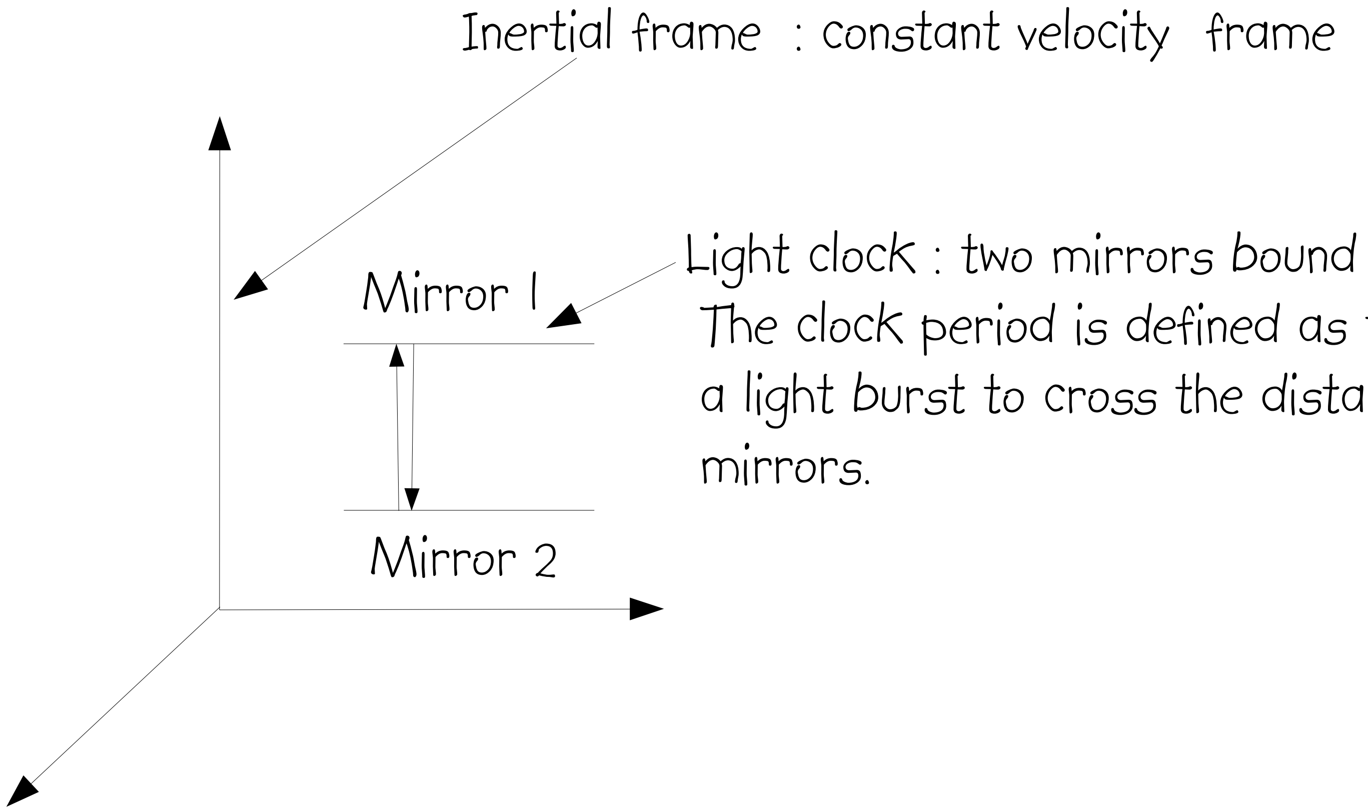
Principles of special relativity

This is a reminder of the two important physical principles introduced in lecture 1.

The relativity principle :
All physical laws are the same in all inertial reference systems.

The universality of the limit velocity :
The modulus of the limit velocity in vacuum observed in an inertial reference system
is universal and does not depend upon the state of motion of the source nor the observer.

Inertial (Galilean) reference system IRS equipped with a light clock :



Light clock : two mirrors bound to the inertial frame. The clock period is defined as the time taken by a light burst to cross the distance separating the mirrors.

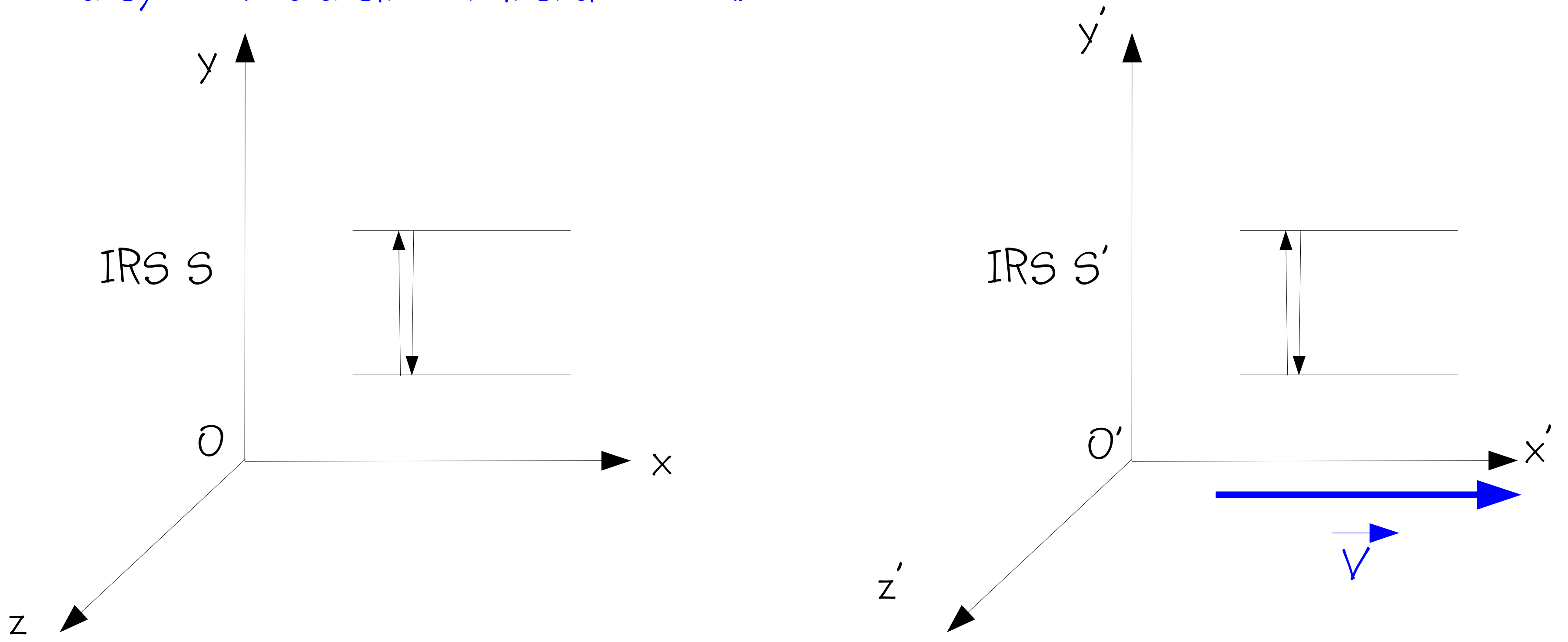
Time dilation :

S' moves with respect to S with a constant velocity V directed towards (0,x)

Their clocks are synchronized when O and O' coincide

By construction, the clocks are made the same.

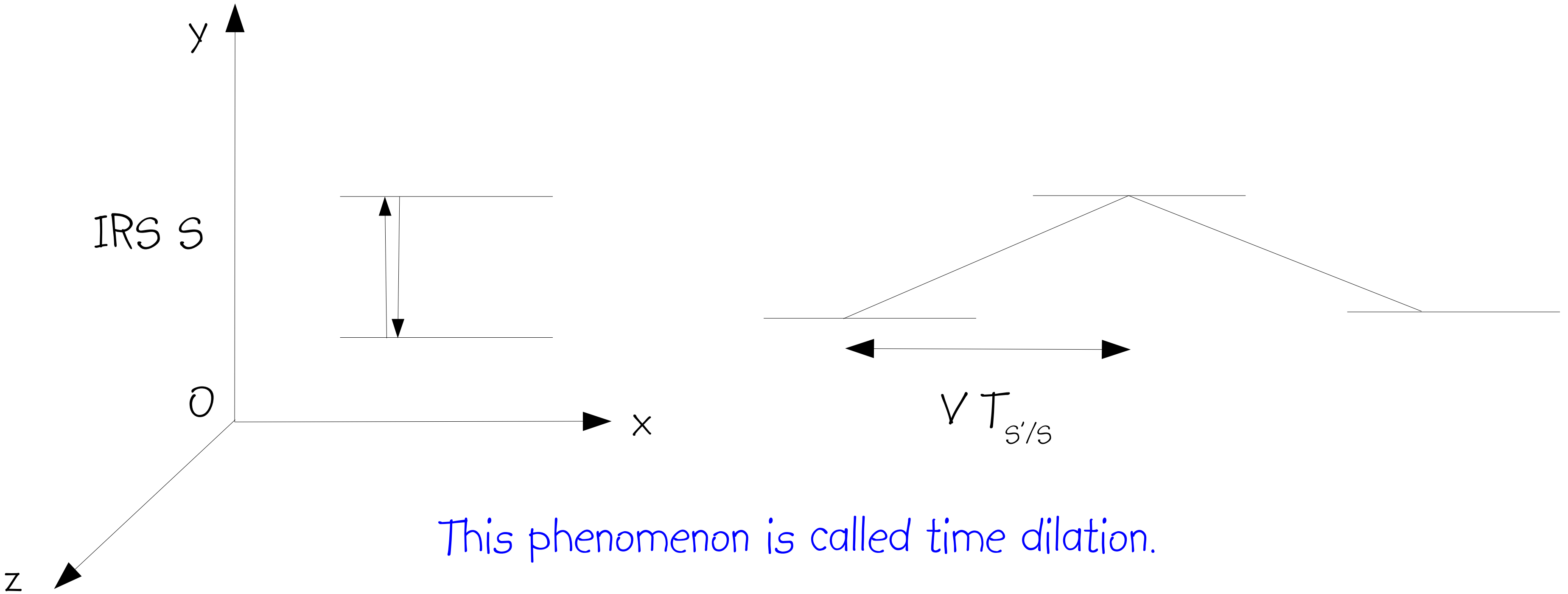
Because of the principle of relativity, both S and S' observe the same time period when they look at their own inertial clock.



Time dilation :

Looking at the clock of S', S observes that the S' mirrors move between two ticks.

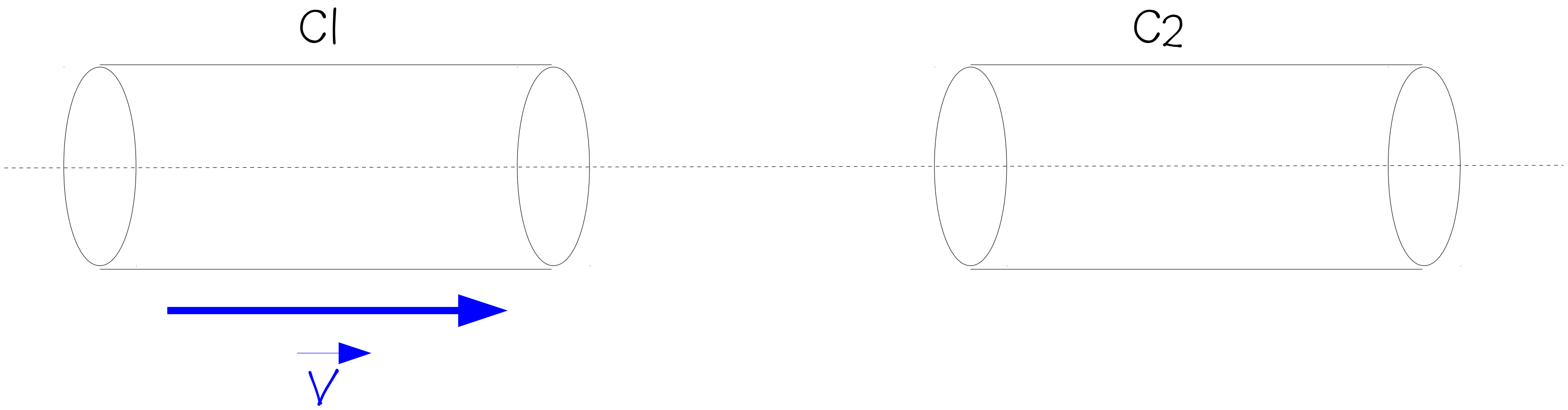
The mirrors of S' observed by S move by a constant distance between two ticks.
As c is universal (second principle) and the total distance travelled by the S' light burst appears bigger to S, S concludes that the S' clock period seems bigger.
Let's call $T_{S'/S}$ the S' clock period as observed by S.



This phenomenon is called time dilation.

Invariance of transverse dimensions :

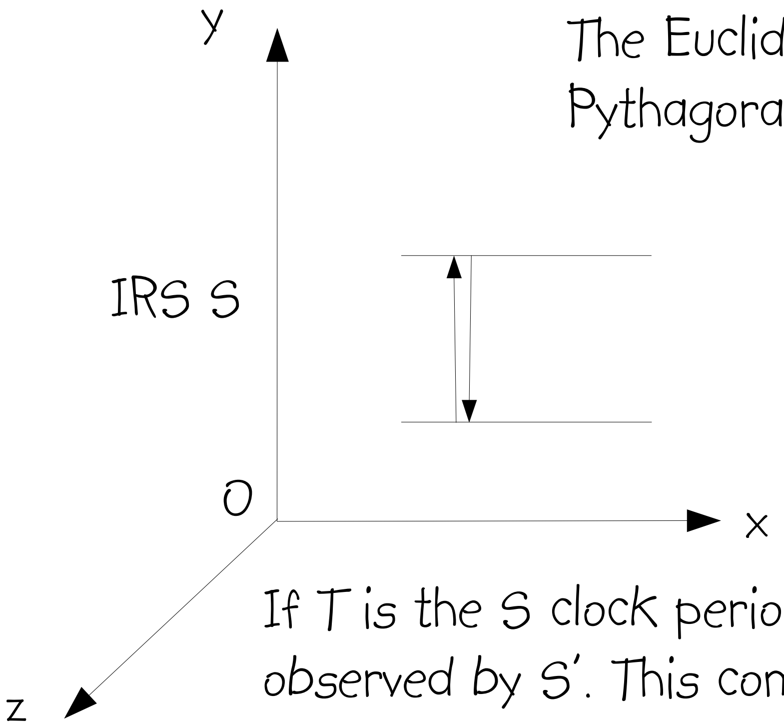
Before quantifying the time dilation factor, let's have a look at the transformation of transverse distances. We examine the motion of two identical cylinders C1 and C2, moving with a constant relative velocity V along their axis.



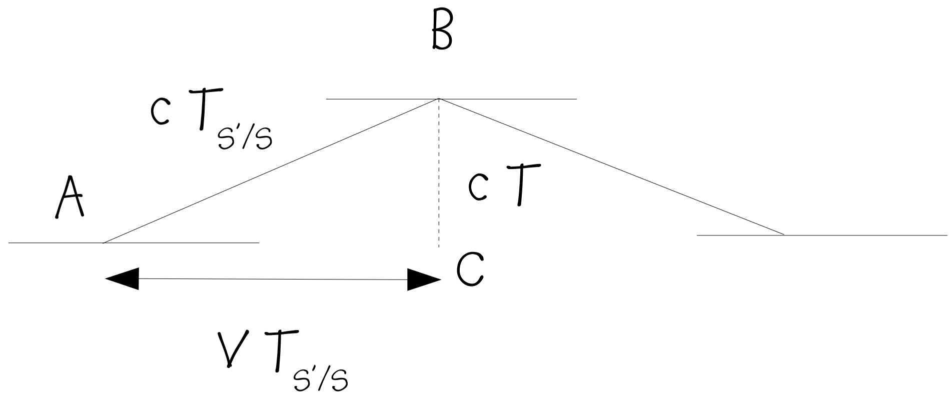
C2 sees C1 moving at the velocity V. Suppose that the diameter of C1 as seen by C2 would appear smaller. Then C1 would slide into C2. However, C1 sees C2 moving in its direction at the velocity V. So C1 should conclude that C2 has a smaller diameter and that C2 should slide into C1. This hypothesis leads to a paradoxical conclusion. The only way out is to consider that both diameters stay the same.

Two IRS measure the same distances in a plane which is transverse to the direction of their relative motion.

Time dilation :



The Euclidean geometry of space remains valid in SR. As a consequence Pythagoras' theorem stands.



If T is the S clock period observed by S, it is also the S' clock period observed by S'. This combined with Pythagoras' theorem leads to :

$$AB^2 = AC^2 + CB^2 = c^2 T_{S'/S}^2 = V^2 T_{S'/S}^2 + c^2 T^2 \;\Rightarrow\; c^2 T_{S'/S}^2 - V^2 T_{S'/S}^2 = c^2 T^2$$

$$T_{S'/S} = \frac{T}{\sqrt{1 - \frac{V^2}{c^2}}} = \gamma T \qquad \text{with :} \qquad \gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2}$$

As observed by S, the S' clock seems to have a bigger period. Then it appears to be slower. Note that V cannot reach c. Note also that this phenomenon is symmetric : when observed by S', the S clock shows the same dilated period.

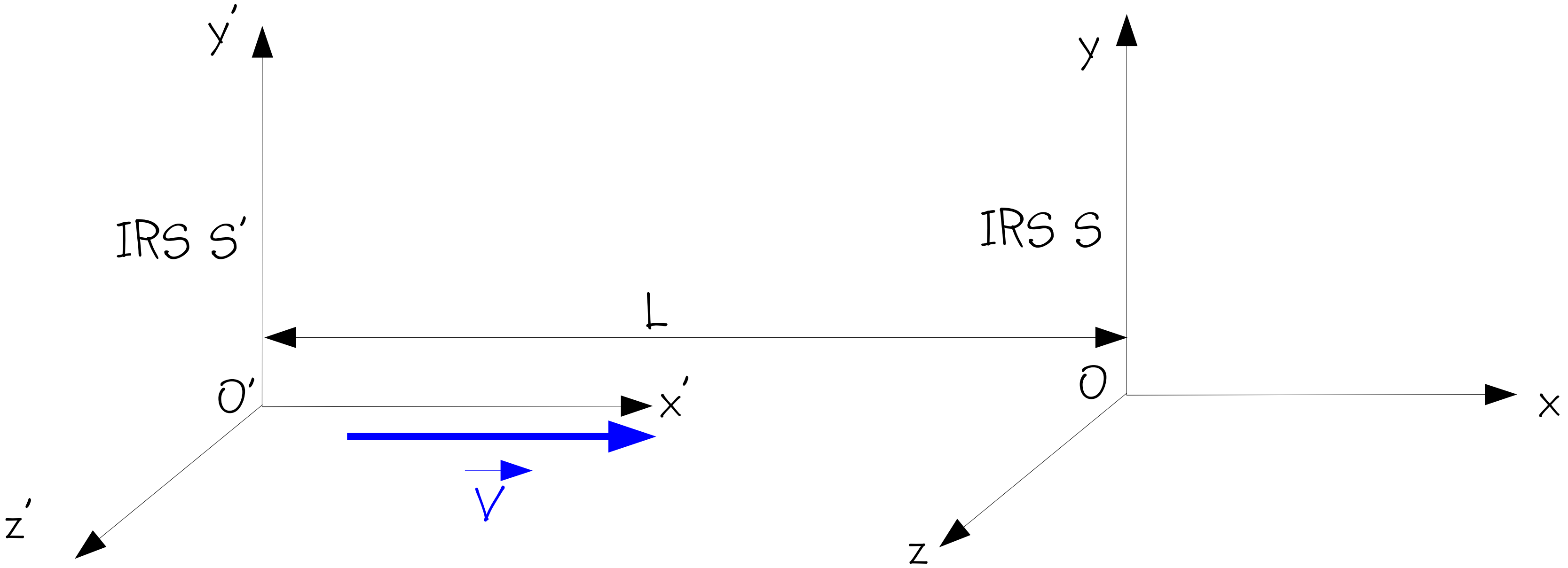
Time dilation

If a physical phenomenon at rest in S lasts a duration t, it also lasts the same duration t when observed at rest in S'.

But if it is observed with a constant relative velocity V, its duration will appear to be dilated by a factor γ .

Then a short-lived particle (like a muon) may appear to live much longer than its natural lifetime measured at rest, if it is observed travelling at fast velocity.

Longitudinal length contraction



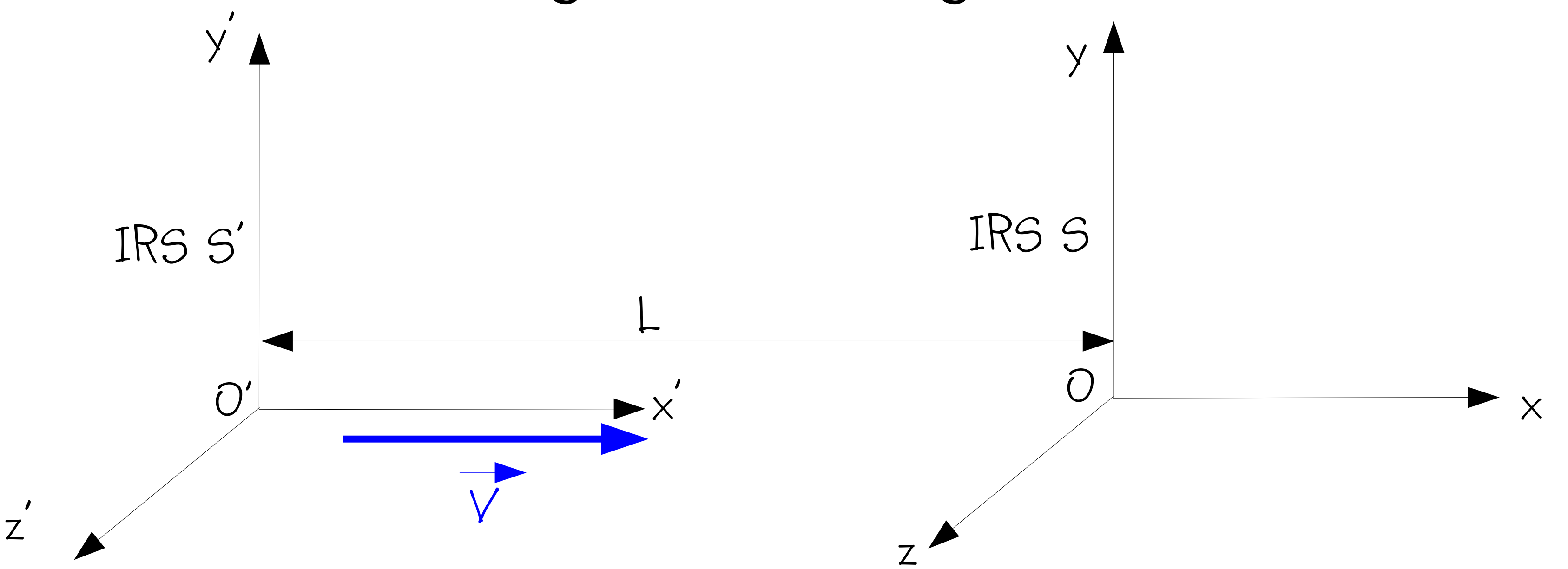
Let's assume two IRS S and S', with S' moving along (o,x) at a constant velocity V.

At a given time the distance between O' and O is measured to be L in S.
If t = L / V is the time taken by O' to reach O when measured in S, then the time measured in S' is : $t' = t\gamma^{-1} = L/V\,\gamma^{-1}$ which is smaller than t.

In S', O also travels towards O' at the velocity V. The distance travelled by O as measured in S' is : $L' = t'V = \frac{L}{V}\gamma^{-1}V = L\gamma^{-1}$ which is smaller than L.

This phenomenon is called the longitudinal length contraction , sometimes also referred as the Lorentz-FitzGerald contraction.

Longitudinal length contraction



A longitudinal length which is measured at a given time (simultaneous measurement) at rest in S will appear to be contracted to S' by a factor γ^{-1} .
This phenomenon is also symmetric. A longitudinal length which is measured at a given time (simultaneous measurement) at rest in S' will appear to be contracted to S by a factor γ^{-1} .

This may appear contradictory, but it is not the case. The important point is to note that the definition of simultaneity is not the same in S and S'. So it leads to different functional measurement protocols.

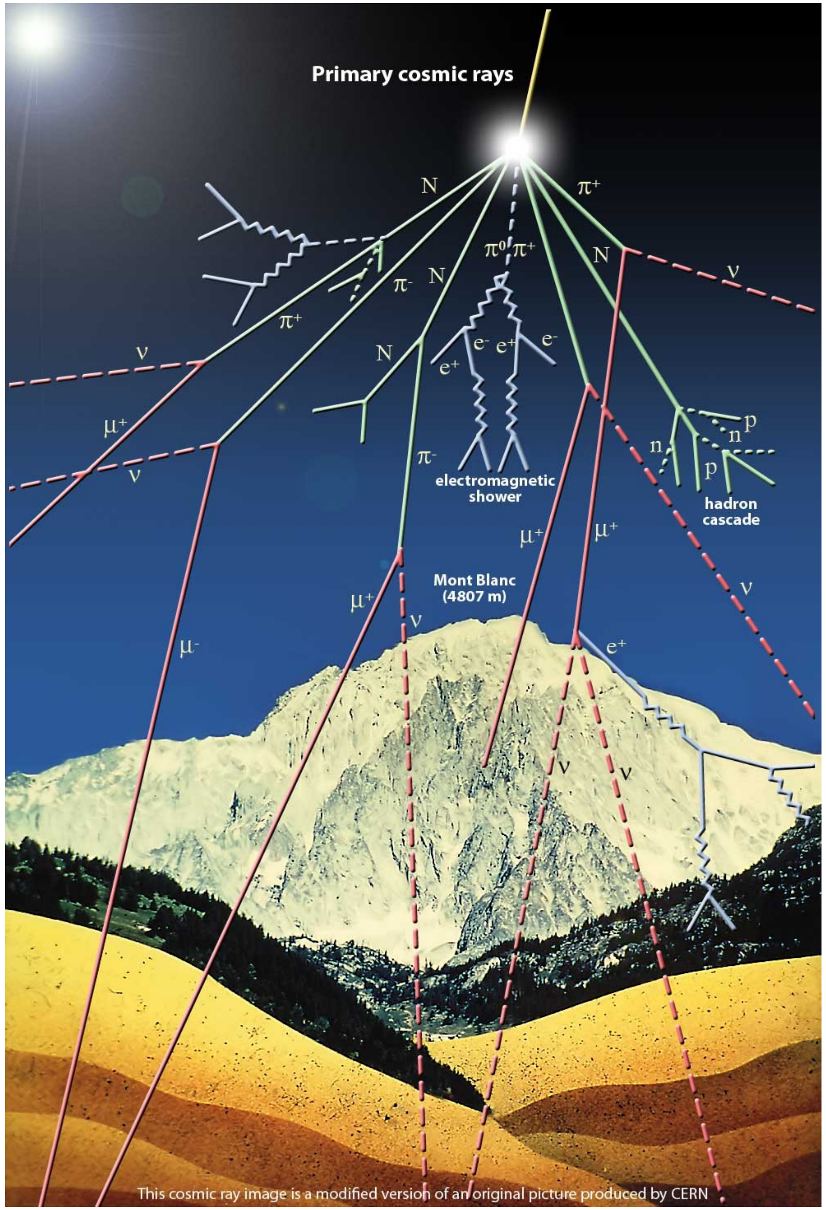
Relativity in every day life

In collisions of cosmic rays that take place in the external layer of the atmosphere (>10 km from the ground), short-lived muons are created. Their lifetime is around 2 μs when measured at rest. Most of these muons carry a big kinetic energy. But even travelling at a speed close to that of light, these muons could only cross around 600 m.

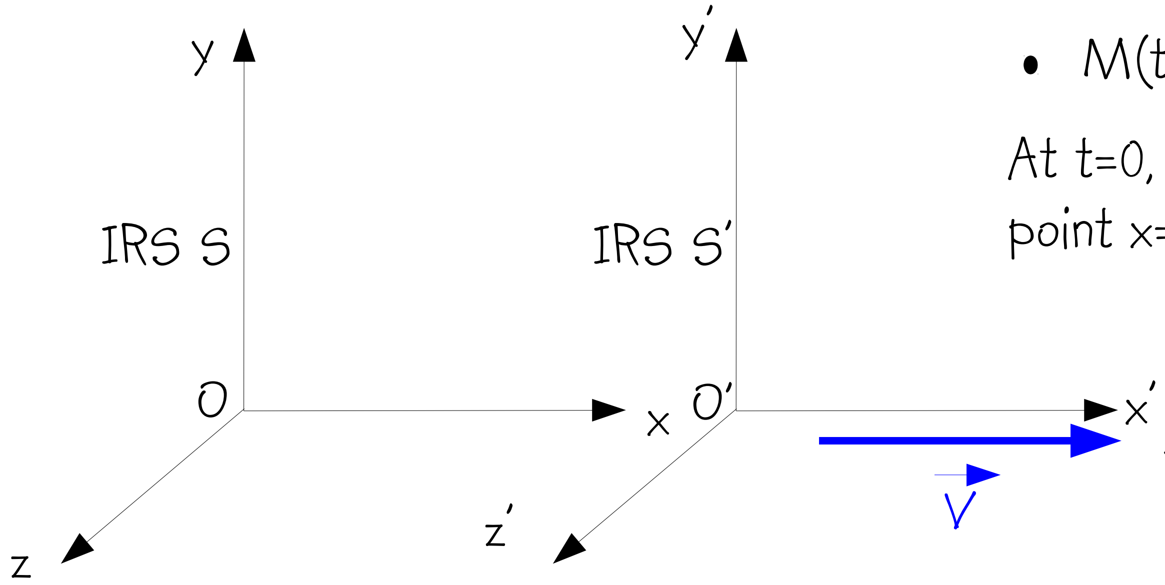
However for an *observer bound* to the earth surface (10 km away), their lifetime is dilated by a big factor that depends upon their travelling velocity. So high-velocity muons reach the ground.

For an *observer bound* to a muon, at rest w.r.t. it, this muon only lives 2 μs. But it now *sees* the atmosphere *being* contracted by a γ factor. So it also reaches the ground.

All in all, *both observers see* the *same* physics phenomenon although their relative perception of time and space differs.



Lorentz transformation :



- M(t,x,y,z) coordinates in S

At t=0, O and O' meet at the same point x=0.

The question is to determine the coordinates of M in S'

Taking into account the longitudinal length contraction, one can write :

$$x' = x\gamma^{-1} - Vt' \quad \text{and reciproqually} \quad x = x'\gamma^{-1} + Vt$$

Which leads to :

$$x'\gamma^{-1} = x\gamma^{-2} - Vt'\gamma^{-1} = x - Vt = x\left(1 - \frac{V^2}{c^2}\right) - Vt'\gamma^{-1}$$

$$-x\frac{V^2}{c^2} + Vt = Vt'\gamma^{-1} \quad \Rightarrow \quad t' = \gamma\left(t - x\frac{V}{c^2}\right)$$

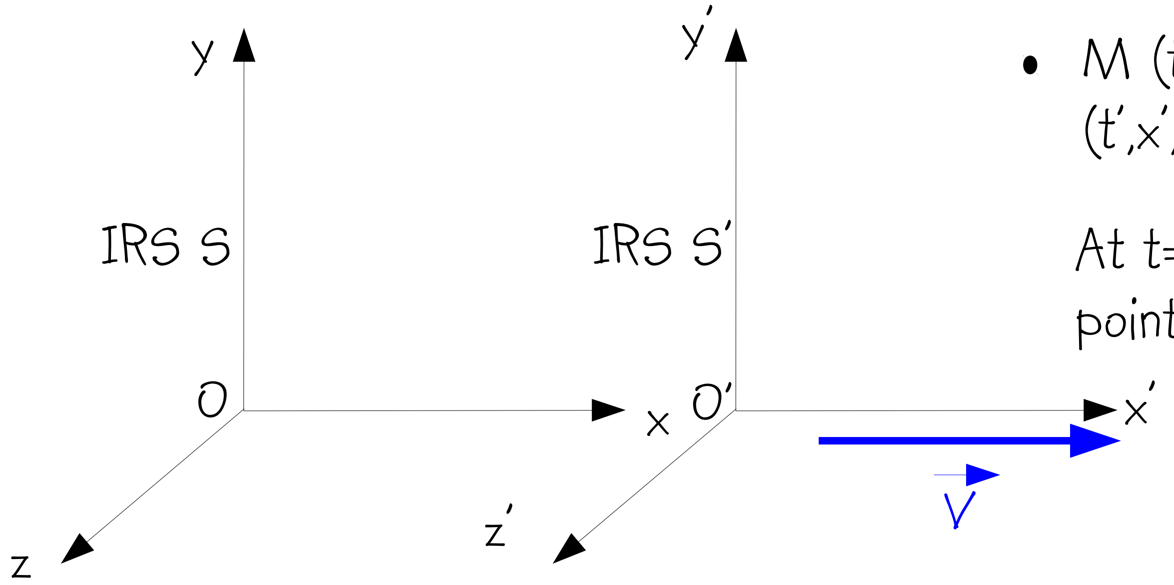
And finally :

$$ct' = \gamma\left(ct - x\frac{V}{c}\right)$$

$$x' = \gamma\left(x - \frac{V}{c}ct\right)$$

z and y , which are transverse dimensions, are not affected by the change of IRS.

Lorentz transformation :



- M (t,x,y,z) coordinates in S
  (t',x',y',z') coordinates in S'

At t=0, O and O' meet at the same point x=0.

After defining the relativistic coefficients : $\beta=\frac{V}{c}$ $\gamma=\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}=(1-\beta^2)^{-1/2}$

The Lorentz transformation in Cartesian coordinates is given by :

$$c\,t'=\gamma(c\,t-\beta\,x)$$
$$x'=\gamma(x-\beta\,ct)$$
$$y'=y$$
$$z'=z$$

It acts symmetrically with respect to ct and x
It mixes time (ct) and space.
A space rotation mixes space coordinates.
A Lorentz transformation mixes space and time coordinates.

Lorentz transformation / Galilean transformation

At low velocity, when V is very small with respect to c :

$$\beta = \frac{V}{c} \simeq 0 \qquad \gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \simeq 1$$

If M is not too far from O, then the Lorentz transformation reduces to a Galilean transformation :

$$t' = t$$

$$x' = x - V\,t$$

$$y' = y$$

$$z' = z$$

Special Relativity reduces to classical mechanics at low velocities

Relativistic addition of velocities

Let's assume that M moves at a constant velocity U towards (o,x') in S' .

Then : $x'=U t'$

$$c t=\gamma(c t'+\beta U t')$$
$$x=\gamma(U t'+\beta ct')$$

Then the velocity of M with respect to S is given by :

$$\omega=\frac{dx}{dt}=\frac{c\gamma(U\,dt'+\beta cdt')}{\gamma(c\,dt'+\beta U\,dt')}=\frac{c(U+V)}{(c+\frac{VU}{c})}=\frac{(U+V)}{(1+\frac{VU}{c^2})}$$

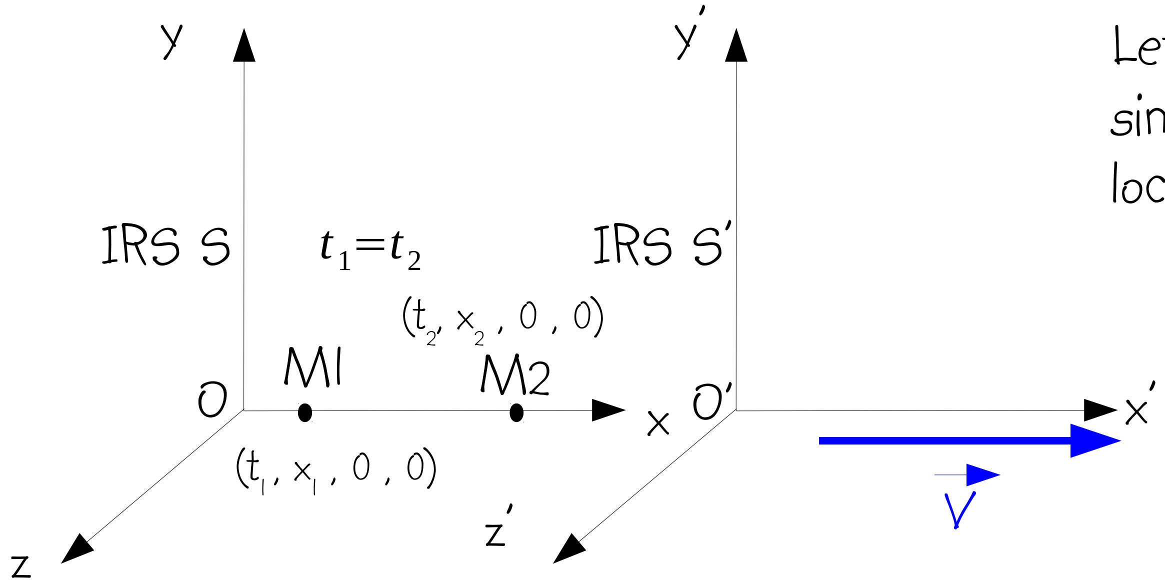
In the low velocity regime : $1+\frac{VU}{c^2}\simeq 1$ then $\omega=U+V$

This is the formula of classical mechanics.

If : $U=c$ then $\omega=c$ in accordance with the limit velocity universality principle.

Exercise : Find the velocity transformation law if M moves transversally at a constant velocity T with respect to S'.
We will assume that the y' coordinate of M in S' is given by y'= T t' . Discuss the limit case when T=c .

Relativistic Simultaneity



Let M1 and M2 be two simultaneous events in S located on the (0,x) axis.

$$c t_1' = \gamma (c t_1 - \beta x_1)$$

$$c t_2' = \gamma (c t_2 - \beta x_2)$$

$$c(t_1' - t_2') = \gamma (c(t_1 - t_2) - \beta (x_1 - x_2)) = \gamma \beta (x_2 - x_1) \neq 0$$

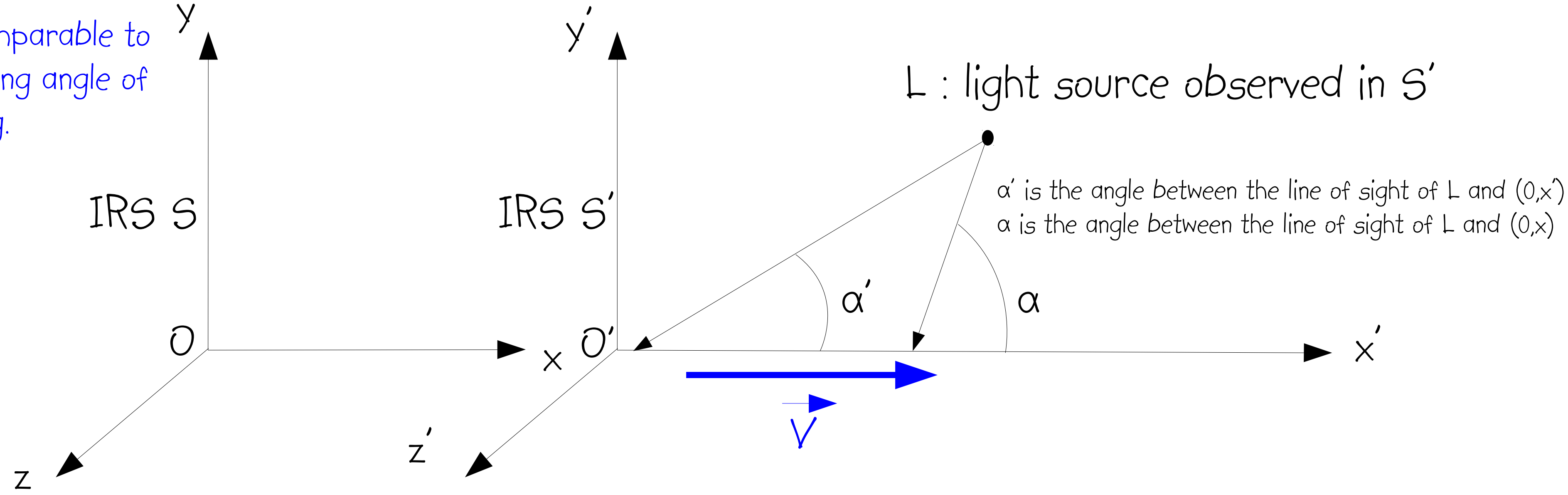
Two simultaneous events in S occur at different times in S'. The simultaneity is now defined with respect to a particular IRS.

The way to understand this is the following : suppose an observer sits at rest at the middle of a rigid bar. If two light signals are simultaneously emitted from each bar end, they reach the observer at the same time. They are then simultaneous. Now suppose that the observer is moving at a constant velocity towards one bar edge. While the light signals propagate, the observer gets closer to one edge and further away from the other. Then the light signals do not reach the observer at the same time anymore. The existence of a limit velocity is what explains the loss of universal simultaneity.

At low speed if : $\gamma \beta (x_2 - x_1) \simeq 0$ simultaneity is restored in both IRS

Relativistic light aberration :

This effect is comparable to the apparent tilting angle of rain when running.



x' component of light ray velocity in S' : $U = -c\cos\alpha'$

x component of light ray velocity in S : $U = -c\cos\alpha$

Using the law of velocity composition :

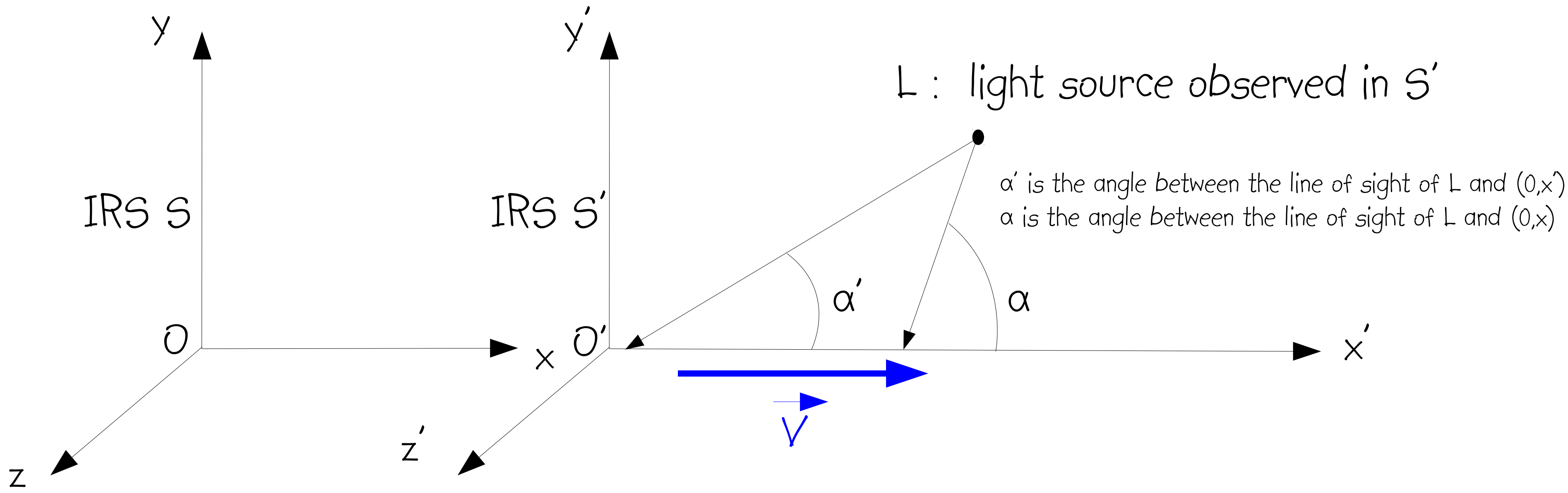
$$-c\cos\alpha = \frac{(-c\cos\alpha' + V)}{(1 - \frac{c\cos\alpha' V}{c^2})} \Rightarrow \cos\alpha = \frac{\cos\alpha' - \beta}{1 - \beta\cos\alpha'}$$

And : $$\cos\theta = \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2}$$

we finally obtain :

$$\tan\frac{\alpha}{2} = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} \tan\frac{\alpha'}{2}$$

Relativistic light aberration :



$$\tan\frac{\alpha}{2}=\frac{\sqrt{1+\beta}}{\sqrt{1-\beta}}\tan\frac{\alpha'}{2}$$

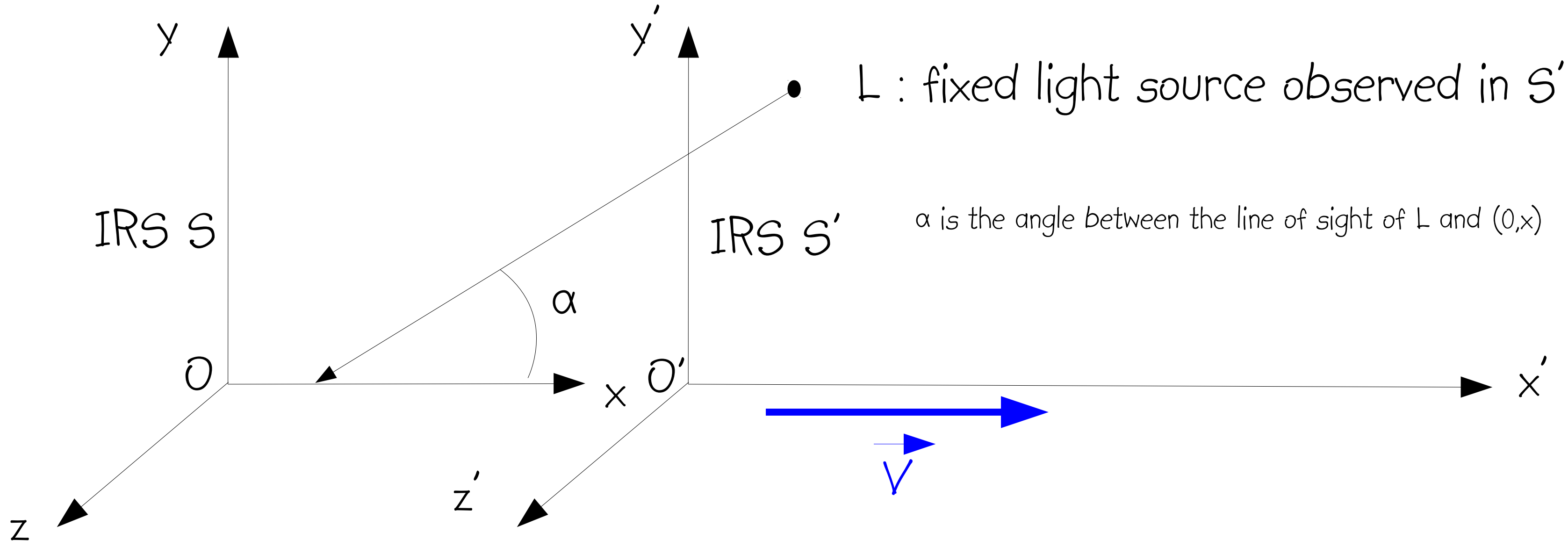
if $\beta > 0$ then $\alpha > \alpha'$ (O moves away from the light source)

if $\beta < 0$ then $\alpha < \alpha'$ (O moves towards the light source)

Relativistic aspect of celestial sphere : Let's consider an observer O sitting in a spaceship and heading towards a particular star S. If S' is the opposite star on the celestial sphere, then as the spaceship gains speed, O will progressively see the view angle of stars getting more and more acute in the direction of S and vice versa in the direction of S'. O will have the impression to see stars moving along meridian lines from S' to S on the celestial sphere, as if getting away while it is in fact getting closer.

When travelling at a velocity close to the speed of light, all visible light rays seem to come straight ahead.

Relativistic Doppler effect :



The L light period as observed in S is increased by two effects : the time dilation plus the the additional length that the light has to travel during a period.

$T = T'\gamma(1+\beta\cos\alpha)$ or expressed using light frequencies : $\nu = \dfrac{\nu'}{\gamma(1+\beta\cos\alpha)}$

In the longitudinal Doppler effect ($\alpha = 0$ or π), the frequency is diminished when the light source is moving away from the observer and vice versa, it is increased when it's heading towards the observer. When the light source moves towards the observer the light is blue-shifted. It is red-shifted in the opposite situation.

In the transverse Doppler effect ($\alpha = \pi/2$), the light is always red-shifted by a factor $1/\gamma$.

Home Work

-Carry out the exercise of page 15

GPS clock :

-using the laws of classical mechanics, find the approximate orbiting velocity of the GPS satellites ?
-what is the relative GPS clock period difference with respect to the period of a terrestrial clock of the same type ?
-what would be the time readings difference after one day ?
-what is the consequence of these results ?

To learn more :

- Modern Physics for Scientists & Engineers, Stephen Thornton & Andrew Rex

- Special Relativity : A first encounter , Domenico Giulini, Oxford University Press

- Introduction à la Relativité : Johann Collot, http://lpsc.in2p3.fr/collot

- Relativité restreinte, Claude Semay et Bernard Silvestre-Brac, Dunod

- The principle of relativity, Dover, New York

- Albert Einstein's special theory of relativity, A. Miller, Springer

- On the Shoulders of Giants: The Great Works of Physics and Astronomy, edited with commentary by Stephen Hawking

- The Feynman lectures on physics, volume 1 , Addison-Wesley